Note on the Drag Coefficient for a Sphere

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We previously described (*1*) a method for calculating upper and lower bounds to the drag coefficient for a sphere falling slowly through an unbounded Ellis model fluid. Two sets of experimental data (*2, 3*) were used to check the calculations.

A third set of data, by Dallon (*4*), has come to our attention. Dallon determined the Ellis model parameters for aqueous solutions of carboxymethyl cellulose, hydroxyethyl cellulose, and polyethylene oxide using a concentric cylinder, rotational viscometer. The terminal velocities of small spherical particles falling through the same fluids were measured and the results correlated.

The problem of determining when the creeping flow assumption is valid arose with Dallon's data. We previously assumed (*5, 1*) that, when

$$N_{Re1} \equiv \left(\frac{\rho V_\infty D}{\eta_0} \right) \left(\frac{\rho V_\infty^2}{\tau_{1/2}} \right)^{\alpha-1} < 0.1 \tag{1}$$

inertial effects were negligible. There is no theoretical justification for this choice.

Truesdell (*6*) pointed out that the extra stress tensor for an incompressible Noll simple fluid could be made dimensionless with respect to a characteristic viscosity μ_0 and characteristic time s_0:

$$\mathbf{S} = \frac{\mu_0}{s_0} \mathbf{S}^* \tag{2}$$

This suggests that for steady state flow of such a fluid, Cauchy's first law could be written in terms of dimensionless variables as

$$(\nabla \mathbf{v}^*) \cdot \mathbf{v}^* = -\left(\frac{P_0}{\rho V^2} \right) \nabla P^* + \left(\frac{\mu_0}{\rho s_0 V^2} \right) \text{div } \mathbf{S}^* \tag{3}$$

Here P_0 is a characteristic value of the modified pressure $P \equiv p + \rho\varphi$, where we assume that the external force per unit mass may be represented by $-\nabla\varphi$. For an Ellis model fluid moving past a sphere, we may define $\mu_0/s_0 \equiv \tau_{1/2}$ (an Ellis model parameter) and $V \equiv V_\infty$ (speed of fluid at a large distance from the sphere). Equation (3) suggests that for sufficiently small values of

$$N_{Re2} \equiv \frac{\rho V_\infty^2}{\tau_{1/2}} \tag{4}$$

it seems reasonable to neglect the inertial effects with respect to viscous effects in analyzing flow of an Ellis model fluid phast a sphere.

For an incompressible Noll simple fluid with fading memory, it has been shown that one dimensionless form of Cauchy's first law for a steady state flow is [*7*, Equations (9) and (15); see *8* for corrections]

$$(\nabla \mathbf{v}^*) \cdot \mathbf{v}^* = -\left(\frac{P_0}{\rho V^2} \right) \nabla P^* + \left(\frac{\mu_0}{\rho V L} \right) \text{div} \left[\sum_{q=1}^{n} \left\{ \frac{s_0 V}{L} \right\}^{q-1} \mathbf{G}_q^* + 0 \left(\left\{ \frac{s_0 V}{L} \right\}^n \right) \right] \tag{5}$$

where $\mathbf{G}_q^*$ is a dimensionless polynomial function of the components of the Rivlin-Ericksen tensors. If we define $\mu_0 \equiv \eta_0$ (an Ellis model parameter) and $L \equiv D$ (diameter of sphere), Equation (5) suggests that for sufficiently small values of

$$N_{Re3} \equiv \frac{\rho V_\infty D}{\eta_0} \tag{6}$$

inertial effects may be neglected with respect to viscous effects in considering flow of an Ellis model fluid past a sphere.

We must choose among N_{Re1}, N_{Re2}, and N_{Re3} in establishing an experimental criteria for creeping flow. The average of the calculated upper and lower bounds was compared with the experimental drag coefficients for all of Dallon's data. The average percent error was calculated when each of these dimensionless groups was less than 10^{-2}, 10^{-1}, 1, and 10. For each of these Reynolds numbers, the average percent error increased as the constraint increased. It appears that the best criterion for creeping flow of an incompressible Ellis model fluid past a sphere is†

$$N_{Re3} < 0.1 \tag{7}$$

For this choice, the average error for nineteen points is 10.2%.

By using inequality (7), the average error in predicting Slattery's data (*2*) is 30% for 101 points [compared with 28% for 100 points by using inequality (*1*) as the criterion for creeping flow]. There were eighty-four points common to both comparisons. For Turian's data (*3*), N_{Re1}, N_{Re2}, and N_{Re3} were all less than 10^{-2}; since the agreement between calculation and experiment was excellent, no smaller constraint was investigated.

NOTATION

D = diameter of sphere
L = characteristic length
N_{Re1}, N_{Re2}, N_{Re3} = Reynolds numbers defined by Equations (1), (4), and (6)
p = pressure
P = modified pressure, $P = p + \rho\varphi$
P_0 = characteristic value of modified pressure
$\mathbf{S}$ = extra stress tensor
s_0 = characteristic time of Noll simple fluid (7)
$\mathbf{v}$ = velocity
V = characteristic speed
V_∞ = speed of fluid at a large distance from sphere

Greek Letters

α, η_0, $\tau_{1/2}$ = Ellis model parameters (*1*)
μ_0 = characteristic viscosity of Noll simple fluid (7)
ρ = density
φ = scalar potential used to represent the external force per unit mass, $\mathbf{f} = -\nabla\varphi$

Superscript

* = dimensionless variable

† A comparison of calculated values (*1*) for the drag coefficient with Dallon's experimental values (*4*) has been deposited as document 00781 with the ASIS National Auxiliary Publications Service, c/o CCM Information Sciences, Inc., 22 W. 34th St., New York 10001 and may be obtained for $1.00 for microfiche or $3.00 for photocopies.

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On the Relation Between Complex Viscosity and Steady State Shearing in the Maxwell Orthogonal Rheometer

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Recent analyses of steady flow in the Maxwell Orthogonal Rheometer (MOR) have indicated that one can relate the complex viscosity $\eta = \eta' - i\eta''$ of a substance to measurements made with the MOR (*1, 2*). In particular, Bird and Harris (*2*) have employed an integral model [Bird and Carreau (*3*)] in which the stress $\boldsymbol{\tau}$ at time t is given by

$$\boldsymbol{\tau} = -\int_{-\infty}^{t} m[t-t', II(t')] \left[\left(1+\frac{\epsilon}{2}\right)\overline{\boldsymbol{\Gamma}} - \frac{\epsilon}{2}\boldsymbol{\Gamma}\right] dt' \quad (1)$$

with a memory function

$$m[t-t', II(t')] = \sum_{p=1}^{\infty} \frac{\eta_p}{\lambda^2_{2p}} \frac{e^{-(t-t')/\lambda_{2p}}}{\left[1+\frac{1}{2}\lambda^2_{1p} II(t')\right]} \quad (2)$$

to obtain the following results:

$$\lim_{\psi\to 0}\left[\frac{\tau_{xz}}{-\Omega\psi}\right] = \eta' \quad (3)$$

$$\lim_{\psi\to 0}\left[\frac{\tau_{yz}}{-\Omega\psi}\right] = \eta'' \quad (4)$$

In Equation (1), $\Gamma_{ij} = \delta_{ij} - (\partial x_\alpha'/\partial x_i)(\partial x_\alpha'/\partial x_j)$, $\overline{\Gamma}_{ij} = (\partial x_i/\partial x_\alpha')(\partial x_j/\partial x_\alpha') - \delta_{ij}$, ϵ is a scalar constant, and $II(t') = \dot{\gamma}_{ij}\dot{\gamma}_{ij}$, where $\dot{\gamma}_{ij} = \partial v_j/\partial x_i + \partial v_i/\partial x_j$. Cartesian tensor notation is employed, and repetition of subscripts implies summation over repeated indexes. x_i and x_i' refer to position coordinates at present time t and past time t', respectively. The reader is referred to Figure 1 of reference 2 for a schematic diagram showing relevant parameters for the MOR. Ω is the (constant) angular velocity of the rotating disks, and $\psi = a/b$ is the ratio of lateral displacement (eccentricity) a between disk centers to gap distance b between disks. A rectangular coordinate system is chosen in which y is taken parallel to the projection onto one of the disks of the line connecting the two disk centers, z is along the axis of one of the disks, and x is orthogonal to y and z.

We show below that reexamination of the assumptions inherent in the derivation of Equations (3) and (4) leads to somewhat different conditions for the equivalence of $\tau_{xz}/(-\Omega\psi)$ with η' and $\tau_{yz}/(-\Omega\psi)$ with η''. In addition, some comments are made concerning the application of Tanner's network rupture model (*4, 5*) to flow in the MOR. Finally, we note a correspondence between measurements in transversely superposed steady shear and oscillatory flow, and flow in the MOR.

The arguments presented below are valid for any memory function $m[t-t', II(t')]$ which fulfills the condition

$$m[t-t', II(t')] \cong m(t-t', 0) \quad (5)$$

as $II(t')$ approaches zero. This condition is consistent with the observed behavior of wide classes of polymer solutions and melts and also agrees with predictions made from rather general theories of the constitutive behavior of materials.

Equation (1) under the condition imposed by Equation (5) then becomes

$$\boldsymbol{\tau} \cong -\int_0^{\infty} m(\tau, 0)\left[\left(1+\frac{\epsilon}{2}\right)\overline{\boldsymbol{\Gamma}} - \frac{\epsilon}{2}\boldsymbol{\Gamma}\right] d\tau \quad (6)$$

STRESS COMPONENTS IN THE MOR

By using a constitutive equation of the form of Equation (1), it is a simple matter to show, with the aid of Equation (14) of (*2*), that steady flow in the MOR is described by $II(t') = 2(\Omega\psi)^2$ and stress components

$$\frac{\tau_{xz}}{-\Omega\psi} = \int_0^{\infty} m[\tau, 2(\Omega\psi)^2]\,\frac{\sin(\Omega\tau)}{\Omega}\,d\tau \quad (7)$$

$$\frac{\tau_{yz}}{-\Omega\psi} = \int_0^{\infty} m[\tau, 2(\Omega\psi)^2]\,\frac{[1-\cos(\Omega\tau)]}{\Omega}\,d\tau \quad (8)$$

Note that Equations (7) and (8) are not dependent upon